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*The Sign of Consequence*

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**Abstract:**

The "sign of consequence" is a notation for propositional logic that Peirce invented in 1886 and used at least until 1894. It substituted the "copula of inclusion" which he had been using since 1870.

**Keywords:** Sign of Consequence, Logic, Notation

The copula of inclusion, symbolized by " $\leftarrow$ " (or " $\leftarrow$ " in its cursive form) was used extensively in Peirce's major works on the algebra of logic in 1870-1885 (Peirce 1870, Peirce 1880, Peirce 1885). The copula of inclusion was regarded by Peirce as functionally complete, because all logical operations of the propositional calculus can be defined in terms of inclusion and constant falsehood. It is known that Peirce discovered the functional completeness of the joint denial for Boolean algebra in 1880 (W 4:218-221), re-discovered and proved to be such by H. M. Sheffer in 1913. Why was inclusion so important, then, and why did Peirce prefer the copula of inclusion to the joint denial, given that both were provably functionally complete? The reason for the preference of inclusion was philosophical: according to Peirce, inclusion mirrors inference. Inference, as inclusion, is a *transitive* (if  $a \leftarrow b$  and  $b \leftarrow c$ , then  $a \leftarrow c$ ), *anti-symmetric* (if  $a \leftarrow b$ ,  $b \leftarrow a$  does not hold) and *reflexive* ( $a \leftarrow a$ ) relation, that is, a non-strict partial order. In the 1881 "On The Logic of Number" (Peirce 1881) Peirce calls any such relation a "fundamental relative of quantity" and the systems of objects having a fundamental relation of quantity a "system of quantity" (W 4:299-300). The fundamental relative of quantity is necessary in the construction of an axiomatic base for arithmetic that Peirce presents in that paper (see Shields 2012). But the transitivity and anti-symmetry of the relation of inclusion is important for Peirce because it mirrors the relation of the premises to the conclusion. If logic merely consisted in a blind or syntactic manipulation of signs, any sole sufficient operator would be as good as any other. But logic is the study of inference, and among sole sufficient operators, only inclusion has the same properties (transitivity, anti-symmetry, and reflexivity) that inferential relations have. Thus, "of all the methods in which propositions may be analyzed and analyzed *correctly*, that one which uses the copula of inclusion alone corresponds to the theory of inference" (NEM 4:174, 1898).

In the 1886 "Qualitative Logic," the copula of inclusion is substituted with the sign of consequence:

[...] every qualitative reasoning about individuals may be expressed by the use of one single symbol besides the expressions for the facts whose relations are examined. This symbol must signify the relation of antecedent to consequent. In the form I would propose for it, it takes the shape of a cross placed between antecedent and consequent with a sort of streamer extending over the former. Thus, "If  $a$ , then  $b$ ," would be written

From  $a$ , it follows that if  $b$  then  $c$ ," would be written

"From 'if  $a$ , then  $b$ ,' follows  $c$ ," would be written

(W 5:361)

From the point of view of the truth-functional analysis of the propositional calculus, the copula of inclusion and the sign of consequence are equivalent: just as the copula of inclusion, the sign of consequence is for Peirce a sole sufficient operator. However, Peirce considers the sign of consequence as something more than a mere sign of inclusion. So we read in an unpublished fragment:

Let us examine the relation of inclusion. The use of a triangle of dots  $\triangle$  to signify *therefore* is common at this day. It seems, like other algebraic signs, to be an old punctuation mark adapted to another use. To express inclusion, various signs are used, of which the following are the principal.

A    B	Hamilton.
A ) B	De Morgan. Later, A )) B.
A -< B	Peirce.
A : B	Maccoll
A =(= B	Schröder.

There is something to be said in favor of all of these, even perhaps the last. But I shall now use a different sign

My excuse is that this is not a mere sign of inclusion. ("The Mathematics of Logic," MS 580)

According to the principle of the Ethics of Notation (itself an application of the principle

of the Ethics of Terminology, CP 2.219-26 = EP 2:263-66), a new notation is to be adopted instead of an older and more established one only upon some *justification*. If no justification for a notational change can be provided, then the older and more established form is to be maintained (MS 253, 1903; MS 530, 1904). Peirce's justification for the substitution of the sign of inclusion with the sign of consequence is that the latter is not a mere sign of inclusion. While truth-functionally equivalent to the signs of inclusion used by Peirce and his fellow logicians, the sign of consequence combines in one notational device two functions that in other notations are kept distinct: the sign of consequence is both a sign of a truth-function and a collectional sign.

[A] further notational convention must be introduced. Using parentheses, just as they are used in algebra, as binding signs, we have to distinguish between

and

To do this, we have only to establish the convention that the vinculum, or horizontal line, which forms a part of the sign of consequence is [to] be extended over the whole antecedent, and all possible ambiguity is removed, without the use of parentheses. Thus, we write

and

(MS 559: 8, c. 1894, formulas in Peirce's hand)

In the notation of the sign of inclusion, in order to distinguish  $A \quad (B \quad C)$  from  $(A \quad B) \quad C$  one needs to use parentheses or other conventions. For example, Peirce treats inclusion as left-associative, and thus writes  $x \quad y \quad z$  for  $x \quad (y \quad z)$  (W 5:176, 1885). By contrast, with the sign of consequence this is not necessary: the scope of horizontal bar or vinculum denotes the antecedent in all cases. Peirce is thus justified in claiming that the sign of consequence does not merely correspond to the sign of inclusion. Rather, it corresponds to the sign of inclusion *plus parentheses*, because it fulfills the functions (truth-function and collectional function) that in the former notation were fulfilled by the joint actions of the sign of inclusion and of the parentheses.

Peirce also suggests that the sign of consequence can be "truncated" and read off as a disjunction:

[W]e can cut the sign of consequence into two parts, the cross signifying 'or,' the vinculum 'not'.  
Thus,

may be regarded as

meaning not X or Y is true. [...] this modification of our notation is so vastly more convenient than what we had before, that the student may well ask why I did not adopt it from the beginning. The answer is, that in thus breaking the sign of consequence and inconsequence we shutter all vestiges of the logical origin of the signs of aggregation and composition. Now, I consider the convenience of a logical algebra a very secondary consideration, since it is of no very great importance as a calculus, while it is very important as an instrument of logical analysis ("The Algebra of the Copula", MS 411: 232-233, c. 1894; formulas in Peirce's hand).

The sign of consequence can be "truncated" in two Boolean parts: the vinculum, which expresses negation, and the usual cross, which as in other of Peirce's papers expresses disjunction. We thus have a "notational derivation" from the untruncated form  $\neg \vee$  to the truncated one  $\neg \vee$  - which shows how negation and disjunction can be derived from the conditional without any structural remodeling of the notation. The truncated version may well be easier to manage, and thus preferable from the point of view of the calculus; but the untruncated version is more philosophical, because it brings (truth-functional) analysis to its extreme (sole sufficient operator).

The sign of consequence itself underwent notational transformations. In particular, in summer 1896 Peirce must have realized that by transforming the vinculum of the sign of consequence into an oval, thereby abandoning the mono-dimensional in favor of the two-dimensional arrangement, what one gets is a version of the Entitative Graphs presented in Peirce 1897:  $\neg \vee$  became  $\neg \vee$ . With the transformation of the sign of consequence into the distinctive notation of logical graphs, the oval or cut, the road to Existential Graphs was now open.

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