


Triadic Logic



Robert Lane

The Commens Encyclopedia
The Digital Encyclopedia of Peirce Studies
New Edition

Edited by Mats Bergman and João Queiroz

URL <http://www.commens.org/encyclopedia/article/lane-robert-triadic-logic>
Retrieved 06.02.2023
ISSN 2342-4257
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Abstract:

Peirce was the first logician to define three-valued logical connectives. In 1909, he defined four one-place three-valued connectives and six two-place three-valued connectives, all of which were rediscovered by later logicians. Peirce's motivation was to accommodate within formal logic a specific, narrow range of propositions he took to be neither true nor false, viz. propositions that predicate of a breach in mathematical or temporal continuity one of the properties that is a boundary-property relative to that breach.

Keywords: Triadic Logic

Peirce's Three-valued Connectives

Charles Peirce was the first logician to define logical operators for a many-valued system of logic.¹ In February 1909, on three pages of a notebook in which he recorded his thoughts on logic (MS 339), he defined several three-valued connectives using the truth-table, or matrix, method.² The system of triadic logic that Peirce envisioned employs the values "V", "F", and "L". He interpreted "V" and "F" as "verum" ("true") and "falsum" ("false"), respectively, and he interpreted the third value, "L", as "the limit."

Peirce's work on many-valued logical connectives was first brought to light by Max Fisch and Atwell Turquette (1966). As Fisch and Turquette describe, it had long been thought that Jan Łukasiewicz (1920, 1930) and Emil Post (1921) had developed the first operators for three-valued logic.³ But Peirce is now recognized as the first to use the truth-table method to define three-valued operators. Subsequent to the publication of Fisch and Turquette's paper, the formal aspects of Peirce's three-valued connectives were explored extensively by Turquette (1967, 1969, 1972, 1973, 1976, 1978, 1981/4).

In conducting his triadic experiments, Peirce defined four different one-place connectives and six different two-place connectives. Peirce's three-valued one-place connectives are:

X	\bar{X}	$\frac{0}{X}$	\dot{X}	\acute{X}
V	F	L	F	L
L	L	L	V	F
F	V	L	L	V

As Fisch and Turquette point out, all four of these connectives were rediscovered by later logicians:

Peirce's \bar{X} corresponds to Łukasiewicz's negation Nx , as well as to Halldén's and Körner's negation operators.

Peirce's $\overset{\circ}{X}$ corresponds to Słupecki's "tertium function" Tx .

Peirce's \acute{X} and \grave{X} correspond respectively to Post's negations $\sim_3 X$ and $\sim_2 X$.⁴

Peirce's three-valued two-place connectives are as follows:

Θ	V	L	F
V	V	V	V
L	V	L	L
F	V	L	F

Θ resembles the disjunction operator of two-valued, classical logic, in that " $x \Theta y$ " takes the maximum of the values taken by " x " and " y " ($V > L > F$). Θ corresponds to Emil Post's (1921) "alternation," V_3 , and to Körner's disjunction (e.g., 1966, p. 39).

Z	V	L	F
V	V	L	F
L	L	L	F
F	F	F	F

Z resembles the conjunction operator of two-valued, classical logic, in that " $x Z y$ " takes the minimum of the values taken by " x " and " y " ($V > L > F$). Z corresponds to Körner's conjunction (e.g., 1966, p.39).

Υ	V	L	F
V	V	L	F
L	L	L	F
F	F	F	F

Υ is similar to the disjunction operator of two-valued, classical logic, in that " $x \Upsilon y$ " takes the maximum of the values taken by " x " and " y " ($V > L > F$) when each of the two conjuncts has a classical value. The value "L" is "infectious," however, in that when either of the disjuncts takes "L", the formula as a whole takes "L". Υ corresponds to a connective used by Bochvar (1939), to Kleene's weak alternation (1952, pp. 327-336) and to Halldén's disjunction operator (1949).

Ω	V	L	F
V	V	L	F
L	L	L	L
F	F	L	F

Ω is similar to the conjunction operator of two-valued, classical logic, in that " $x \Omega y$ " takes the minimum of the values taken by " x " and " y " ($V > L > F$) when each of the two conjuncts has a classical value. The value "L" is "infectious," however, in that when either of the conjuncts takes "L", the formula as a whole takes "L". Ω corresponds to a connective used by Bochvar (1939), to Kleene's weak conjunction (1952, pp. 327-336), and to Halldén's conjunction operator (1949).

Φ	V	L	F
V	V	V	V
L	V	L	F
F	V	F	F

Ψ	V	L	F
V	V	V	F
L	V	L	F
F	F	F	F

Turquette holds Φ and Ψ to be more mysterious than Peirce's other two-place connectives, since, as he says, "their motivation is not entirely clear and they seem to have played no very important part in later literature on triadic logic" (1967, p. 66). He argues that Peirce may have been motivated to introduce these connectives by considerations of duality and completeness. Parks (1971), on the other hand, points out that Φ and Ψ did in fact play a part in the later development of triadic logic: they occurred as \mathbf{A} and \mathbf{K} (disjunction and conjunction) in the system developed by Sobocinski (1952), and, subsequent to the publication of Turquette (1967), they occurred as \vee and \wedge in the "logic of ordinary discourse" developed by Cooper (1968) as well as in work by Belnap (1970).

Philosophical Motivations

Peirce characterized triadic logic as

that logic which ... recognizes that every proposition, S is P, is either true, or false, or

else S has a lower mode of being such that it can neither be determinately P, nor determinately not-P, but is at the limit between P and not P. (MS 339, Feb. 23, 1909)⁵

In experimenting with many-valued connectives, Peirce was motivated by the desire to accommodate within formal logic propositions which are neither true nor false; and this means that he believed that some propositions are, indeed, neither true nor false. He thus rejected the Principle of Bivalence (PB), according to which any proposition is either true or else false.⁶

Commentators disagree about Peirce's philosophical reasons for rejecting PB and acknowledging propositions that are at "the limit" between true and false. Because of potentially misleading comments Peirce made regarding the principle of excluded middle (PEM) (see Lane, 2001), some have taken him to have intended "L" to value object-general propositions (roughly, universally quantified propositions). This has the odd consequence that, for example, the proposition "All bachelors are unmarried" is neither true nor false. But for Peirce, to say that PEM does not apply to a proposition "S is P" is not to imply that "S is P" is neither true nor false. The non-application of PEM to general propositions did not motivate the development of Peirce's three-valued connectives.

Others have assumed that Peirce meant "L" to be taken by what he called "vague" propositions, presumably because he held that the principle of contradiction (PC) does not apply to such propositions (see Chiasson, 2001; Lane, 2001). By "vague proposition" Peirce meant object-indefinite propositions (roughly, existentially quantified propositions). So the view that "L" values vague propositions has the odd consequence that, for example, the proposition "Some US President is from Texas" is both true and false. But for Peirce, to say that PC does not apply to a proposition "S is P" is not to imply that "S is P" is both true and false. The non-application of PC to vague propositions did not motivate the development of Peirce's three-valued connectives.

Still others have assumed that Peirce intended his third value to be taken by modal propositions. This is because Peirce wrote that PEM does not apply to assertions of necessity and PC does not apply to assertions of possibility. But a correct understanding of Peirce's "principles of excluded middle and contradiction" shows that these comments do not suggest that a value other than "true" and "false" is needed for modal propositions. Considerations from quantum physics have led others to suggest that Peirce intended "L" to be taken by any proposition containing "scientifically sound predicates" (Jauhari, 1985), including propositions that express natural laws. But this

interpretation is not at all supported by the textual evidence.

In fact, Peirce intended his third value to be taken only by propositions that predicate of a breach in mathematical or temporal continuity one of the properties that is a boundary-property relative to that breach. I call such propositions *boundary-propositions*. To see that this is the sort of proposition Peirce intended "L" to value, we must recognize the distinction between saying that a logical principle does not *apply* to a proposition and saying that it is *false* with regard to a proposition. Peirce's view is that a principle can only be false with regard to a proposition if it applies to that proposition. (MS 641:24 2/3 - 3/4, 1909) So to say that a given principle does not apply to a proposition is to imply that the principle *is not* false with regard to that proposition.

This distinction is important to a correct understanding of Peirce's triadic logic because he intended his triadic logic to accommodate propositions with regard to which PEM is false, and thus to which PEM *applies*. This means, first, that L-propositions (propositions that take Peirce's third value, "L") have individual (non-general) subject-terms. It also means that L-propositions do not express necessity, i.e., they are not of the form "S must be P". Further, Peirce's view seems to have been that PC is true with regard to L-propositions; and this means that PC applies to L-propositions, and therefore that L-propositions have definite (non-vague) subject-terms and that they do not express possibility, i.e., they are not of the form "S may be P" or "S can be P". In sum: L-propositions have singular (individual and definite) subject-terms and are non-modal (they express neither necessity nor possibility).

On one of the pages of the logic notebook in which he defined his three-valued connectives, Peirce gave an example involving an ink-blot. He seems to have intended that example as an illustration of an object-singular, non-modal proposition that takes "L" as its value:

Thus, a blot is made on the sheet. Then every point of the sheet is unblackened or is blackened. But there are points on the boundary line, and those points are insusceptible of being unblackened or of being blackened, since these predicates refer to the area about S and a line has no area about any point of it. (MS 339, February 23, 1909)

The question Peirce found interesting was whether the boundary between the ink blot and the rest of the paper is black or non-black. His answer, it seems, was "*neither*." Again, Peirce described an L-proposition "S is P" as follows:

S has a lower mode of being such that it can neither be determinately P, nor determinately not-P, but is at the limit between P and not P. (MS 339, February 23, 1909)

The boundary between the black ink blot and the non-black paper is neither black nor non-black, and the (object-singular, non-modal) propositions "The boundary is black" and "The boundary is non-black" are neither true nor false. Each is the sort of proposition that Peirce thought should take the value "L". The boundary between the black and the non-black areas of the paper is a *continuity-breach*; it is a line in an otherwise uninterrupted surface. Peirce intended "L" to value propositions that predicate of a mathematical or temporal continuity-breach one of the properties that is a boundary-property relative to that breach. Such propositions are *boundary-propositions*.

This might seem strange at first. Why, after all, would Peirce take boundary-propositions to be interesting or important enough to motivate him to introduce three-valued connectives? The answer lies in the fact that the notion of continuity was itself of supreme philosophical importance for Peirce. That the question of continuity-breaches and their boundary-properties was for him not simply an afterthought or a relatively unimportant aspect of the broader issue of the nature of continuity, is indicated by the fact that each time he revised his definition of continuity in a significant way, his position regarding continuity-breaches and their boundary-properties changed as well. (Lane 1999)

Blocking the way of inquiry?

What are the consequences of Peirce's rejection of PB for his pragmatic account of truth, i.e., his account of truth as that which would be agreed upon at the hypothetical, ideal limit of inquiry? As noted by Cheryl Misak (1991), Peirce did not intend (at least, not from the 1890s on) to give a biconditional definition of "truth" but instead held that what she calls the Truth to Inquiry Conditional is a regulative principle of inquiry, a hope that must be adopted by an inquirer with regard to the question she is investigating:

The Truth to Inquiry Conditional: If "S is P" is true, then, if inquiry relevant to whether S is P were pursued as far as it could fruitfully go, it would be agreed that S is P.

Peirce's view of bivalence seems to have been the same. So in rejecting bivalence with regard to a proposition "S is P", Peirce was in effect giving up the hope that, if inquiry with regard to whether S is P were pursued as far as it could fruitfully go, belief about whether S is P would never be settled.

This seems to be in tension with Peirce's injunction against blocking the "way of inquiry" (CP 1.135, c.1898)⁷; after all, one way to block the way of inquiry is to assert,

with regard to a given question, that inquiry would never result in consensus regarding the answer to that question. Had he claimed that a broad class of proposition (modal propositions, say, or, propositions containing "scientifically sound predicates") fails to be either true or false, Peirce himself would have been guilty of blocking a relatively wide avenue of inquiry. But he rejected bivalence only for a very narrow range of propositions: boundary-propositions. Thus, Peirce was guilty of blocking, not a wide avenue of inquiry, but only a narrow alley-way.

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Endnotes

1. Here I follow standard usage, according to which only n -valued logics ($n > 2$) are "many-valued". See, e.g., Haack (1978, p. 205). ←
2. Peirce himself had originated the truth-table method in 1885, employing it to decide whether a wff is a tautology (CP 3.387). See, e.g., Berry (1952, p. 158). As Fisch and Turquette (1966, pp. 71-72) point out, Łukasiewicz and Tarski (1930, p. 40 n.2) and Church (1956, p. 162) also refer to Peirce in this regard. ←
3. As Fisch and Turquette note, Łukasiewicz and Tarski (1930, p. 40 n.2 and 47 n.2) and Church (1956, p. 162) both refer to Łukasiewicz (1920) and Post (1921) as the originators of three-valued propositional calculus. ←
4. On Łukasiewicz's negation connective, see Łukasiewicz (1930, pp. 47-48), Lewis and Langford (1959, pp. 213-214), and Rescher (1969, pp. 22-23). On Halldén's negation

- operator, see Halldén (1949). On Körner's negation operator, see, e.g., Körner (1966). On Ślupecki's "tertium function," see Ślupecki (1936, pp. 9-11) and Rosser and Turquette (1952, ch.2). On Post's negation connectives, see Post (1921) and Rescher (1969, p. 53). ↵
5. The page of Peirce's logic notebook (MS 339) from which this quotation is taken, as well as the other two pages which record Peirce's work in triadic logic, are reproduced in Fisch and Turquette (1966, pp. 73-75). ↵
 6. Cf. Turquette's (1983) suggestion that Peirce's "verum" and "falsum" were intended by Peirce to be distinct from "true" and "false"; if this is correct, then Peirce may not have considered the assignment of "L" to a proposition to imply that the proposition is neither true nor false, and thus may not have had in mind that L propositions threaten PB. ↵
 7. "Do not block the way of inquiry" is, on Peirce's view, a corollary of what he calls "the first rule of reason": "in order to learn you must desire to learn, and in so desiring not be satisfied with what you already incline to think" (CP 1.135, c.1898). For a discussion of Peirce's first rule or reason, see Haack (1997). ↵